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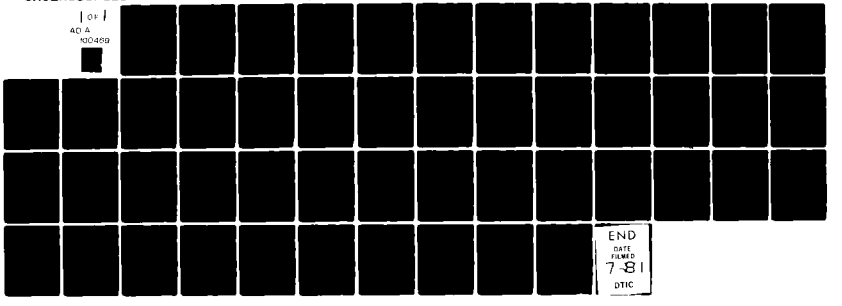

LEVEL II

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STATISTICAL REPRODUCIBILITY OF THE DYNAMIC AND STATIC FATIGUE EXPERIMENTS

by

John E. Ritter, Jr.

Karl Jakus

TECHNICAL REPORT

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Block 20. Abstract (continued)

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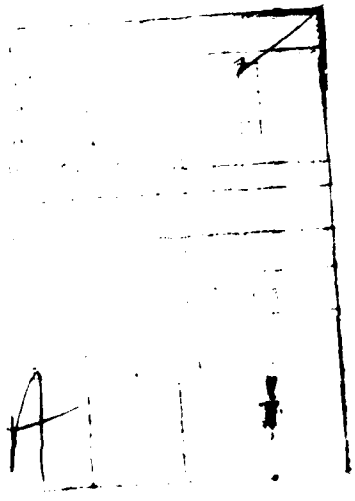

FOREWORD

This report describes the results of an experimental program oriented toward a better understanding of lifetime predictions for optical glass fibers. Some of the progress made toward this goal is summarized in the attached technical paper comprising this report.

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STATISTICAL REPRODUCIBILITY OF THE DYNAMIC AND STATIC FATIGUE EXPERIMENTS

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ABSTRACT

The number of test samples used to characterize the fatigue constants needed for failure predictions for ceramic materials determines the confidence in these predictions. The statistical reproducibility of the dynamic and static fatigue experiments used to measure the fatigue constants was analyzed using both statistical theory and a Monte Carlo computer simulation technique. It was found that the statistical reproducibility depended not only on the number of test samples but also on the other experimental test variables. It was shown that the uncertainty in the statistical reproducibility can be large especially for sample size less than about 100. Guidelines for selecting the optimum sample size for a given dynamic or static fatigue experiment are given. It is recommended that before meaningful conclusions can be drawn regarding the effect of a test variable on fatigue, the statistical reproducibility of the experiment be determined.

1.0 INTRODUCTION

Methods of dealing with design problems involving fatigue of ceramic materials have been developed over the past 10 years through the application of fracture mechanics principles. Since these principles can be used to characterize both the conditions for subcritical crack growth and the conditions for crack instability, they can be used for purposes of design to estimate the allowable stress, or the expected lifetime, or the proof stress necessary to assure a minimum lifetime. This is accomplished by estimating the initial crack size in a ceramic component and the time required for this initial crack to grow to a critical size for spontaneous fracture. For example, it has been derived by assuming a simple power law relationship between subcritical crack velocity and stress intensity that the failure time (t_f) under a constant applied stress (σ_a) is:[1,2]

$$t_f = B S_i^{N-2} \sigma_a^{-N} \qquad (1)$$

where $B = 2/(AY^2 (N-2) K_{IC}^{N-2})$, A, N = material/environment constants, Y = geometric constant (about 1.2 for surface flaws), K_{IC} = critical stress intensity factor, and S_i = fracture strength in an inert environment. B and N in Eq. (1) are fatigue constants that for a given material/environment system characterize subcritical crack growth. The inert strength in Eq. (1) characterizes the initial flaw size. If proof testing is used to truncate the flaw distribution, then the minimum inert strength after proof testing is equal to the maximum proof stress (σ_p), hence, the correspondingly minimum failure time (t_{min}) is:[1,2]

$$t_{min} = B \sigma_p^{N-2} \sigma_a^{-N} \quad (2)$$

From Eq. (1) and (2) it is seen that failure predictions are dependent on the fatigue parameters N and B. These parameters are constants for a given material/environment system and can be experimentally determined directly using fracture mechanics techniques,[3] or can be indirectly measured using static fatigue or dynamic fatigue experiments.[2] Unfortunately, failure predictions are extremely sensitive to the experimental uncertainty in the fatigue parameters. Statistical techniques for estimating this uncertainty in failure predictions have been developed;[4,5] however, the statistical reproducibility of the experimental techniques used to evaluate the fatigue parameters N and B has not been previously determined. Statistical reproducibility is due to random sampling errors that are inherent in every experiment. In the random selection of a finite number of samples for testing, one would expect to see some statistical variability in the measured properties that would be dependent on the number of samples selected. It further would be expected that this variability in the estimation of the fatigue parameters would increase as sample size decreases; however, this statistical reproducibility has never been quantified although some Monte Carlo results on the reproducibility of N as determined by dynamic fatigue tests have been previously reported by the present authors.[6] Before meaningful conclusions can be drawn from the results of fatigue experiments, the statistical reproducibility of these experiments must be known.

The purpose of this paper is to quantify the statistical reproducibility of the dynamic and static fatigue experiments for measuring the fatigue constants N and B. Statistical reproducibility is analyzed using both

statistical theory and a Monte Carlo computer simulation technique. Since the statistical theory contains a number of critical assumptions, it is important to independently validate the statistical theory approach with the Monte Carlo technique. Emphasis is placed on the dynamic and static fatigue techniques because they are increasingly being used to measure the fatigue behavior of ceramics. This is because these test techniques can utilize samples containing flaws representative of those on actual components. It is believed that the results of this study will lead to a better understanding of the static and dynamic fatigue test techniques and their statistical variability. From this information guidelines for sample size requirements in terms of optimum statistical reproducibility can be developed.

2.0 ANALYSIS OF STATISTICAL REPRODUCIBILITY

2.1 Dynamic Fatigue

Dynamic fatigue data is generated by measuring the fracture strength of a number of samples at several constant stressing rates. The fatigue constants N and B can be determined from dynamic fatigue data through using one of four analyses: median, homologous stress, iterative bivariant, or iterative trivariant.[2,7] Since all of these techniques analyze the same set of dynamic fatigue data in determining N and B through a linear regression analysis, all are expected to result in essentially the same statistical reproducibility for N and B. Thus, the median analysis technique was chosen for this study because of its simplicity and wide usage.

With the median analysis the dynamic fatigue data are fitted to:[2,7]

$$\ln \hat{S}_a = a_1 \ln \dot{\sigma} + a_2 \tag{3}$$

where $\hat{S}_a$ = median fatigue fracture strength, $\dot{\sigma}$ = stressing rate, and a_1, a_2 = linear regression constants. The fatigue constants are then determined from:[2,7]

$$N = \frac{1}{a_1} - 1 \tag{4a}$$

$$\ln B = (N+1)a_2 - \ln(N+1) - (N-1)\ln \hat{S}_i$$

$$= a_2 b_2 + \ln a_1 - \left(\frac{1}{a_1} - 3\right)\ln \hat{S}_i \tag{4b}$$

where $\hat{S}_i$ = median inert strength.

Dynamic fatigue strength data of an "ideal" material was simulated on a computer using a Monte Carlo technique.[8,9] It was assumed that the fatigue constants N and B of this ideal material are given and that the inert strength distribution is given by a two parameter Weibull distribution[10]

whose slope and scale parameters, m_i and S_{0i} respectively, are known. With the Monte Carlo technique a given number of samples at a specific stressing rate were chosen by randomly selecting their failure probability from a uniform distribution between 0 and 1. The corresponding fatigue fracture strengths were then calculated according to the fracture mechanics relationship:[2]

$$\ln S_f = \frac{1}{N+1}\left[\ln B(N+1) + (N-2)\left(\frac{1}{m_i}\ln\ln\frac{1}{1-F} + \ln S_{0i}\right) + \ln\dot{\sigma}\right] \tag{5}$$

A separate set of inert strength samples were chosen similarly to the fatigue samples, namely failure probabilities were randomly chosen and the corresponding inert strength were calculated from the two parameter Weibull distribution:[10]

$$\ln S_i = \frac{1}{m_i}\left(\ln\ln\frac{1}{1-F}\right) + \ln S_{0i} \tag{6}$$

Once a set of inert strength and fatigue strengths at several different stressing rates were randomly chosen, N and B were determined from this data using Eqs. (3) and (4). It should be noted that the median strength for a given set of strength values was determined by ranking and fitting the strengths to a Weibull distribution by linear least square method. The median strength was then calculated as the value at $F = 0.50$. Alternatively, the median strength could have been determined by choosing the actual median strength value; however, this results in greater variability and thus was not used. By iterating this procedure 100 times, distributions for N and B were generated which represent the statistical reproducibility of N and B as determined from the dynamic fatigue test. With the 100 values generated by the Monte Carlo technique, the average values with their corresponding variances and the covariance between N and B were calculated from the usual statistical formulas:[11]

$$\bar{N} = \frac{1}{n} \sum_{j=1}^{n} N_j \tag{7a}$$

$$\overline{\ln B} = \frac{1}{n} \sum_{j=1}^{n} \ln B_j \tag{7b}$$

$$V(N) = \frac{1}{n-1} \sum_{j=1}^{n} (N_j - \bar{N})^2 \tag{7c}$$

$$V(\ln B) = \frac{1}{n-1} \sum_{j=1}^{n} (\ln B_j - \overline{\ln B})^2 \tag{7d}$$

$$Cov(N, \ln B) = \frac{1}{n-1} \sum_{j=1}^{n} (N_j - \bar{N})(\ln B_j - \overline{\ln B}) \tag{7e}$$

where n = 100 for these computer calculations. For a given material/environment system, i.e. for a given set of m, S_0, N, and B, the important test variables studied were the number of samples per stressing rate, the stressing rate range, and the number of stressing rates that were used in a given test. For the case of multi-stressing rates, the stressing rates were evenly spaced from maximum to minimum. Figure 1 gives a schematic flow diagram of the Monte Carlo computer simulation technique for determining the fatigue constants N and B by the dynamic fatigue test. Note that the same number of inert samples were chosen as the number of samples per stressing rate.

The variances and covariance of the inert strength parameters were also derived as a function of sample size from statistical theory by making certain assumptions (see Appendix, Eqs. (A16), (A26), and (A27)). The results of this analysis are as follows:

$$V(m_i) \simeq \frac{m_i^2}{J} \qquad (8a)$$

$$V(\ln S_0) \simeq \frac{1}{J}\left\{\frac{\Gamma(1+2/m_i) - \Gamma^2(1+1/m_i)}{\Gamma^2(1+1/m_i)} - \left[\ln\Gamma(1+1/m_i)\right]^2\right\} \qquad (8b)$$

$$Cov(m_i, \ln S_{0i}) \simeq \frac{m_i}{J}\ln\Gamma(1+1/m_i) \qquad (8c)$$

where J = number of samples and Γ = gamma function. Likewise, the variances in the fatigue constants N and B and their covariance were derived as shown in the Appendix, Eqs. (A41), (A42), and (A43), to be:

$$V(N) \simeq \frac{K_d (N-2)^2 (N+1)^2}{J_0 m_i^2 R(\ln\dot{\sigma})} \qquad (9a)$$

$$V(\ln B) \simeq \frac{K_d (N-2)^2}{J_0 m_i^2 R(\ln\dot{\sigma})}\left\{\left[3\ln\hat{S}_i + 1 - \ln B(N+1) - \overline{\ln\dot{\sigma}}\right]^2 - R(\ln\dot{\sigma})(1-\bar{r}_2)\right\} \qquad (9b)$$

$$Cov(N, \ln B) \simeq \frac{K_d (N-2)^2 (N+1)}{J_0 m_i^2 R(\ln\dot{\sigma})}\left[\ln B(N+1) - 3\ln\hat{S}_i - 1 + \overline{\ln\dot{\sigma}}\right] \qquad (9c)$$

where J_0 = total number of samples used in dynamic fatigue experiment, J_2 = number of stressing rates, $R(\ln\dot{\sigma}) = \frac{1}{J_2}\sum_{j=1}^{J_2}(\ln\dot{\sigma}_j - \overline{\ln\dot{\sigma}})^2$, $\overline{\ln\dot{\sigma}} = \frac{1}{J_2}\sum_{j=1}^{J_2}\ln\dot{\sigma}_j$,

$$K_d = m_i^2\left(\frac{N+1}{N-2}\right)^2 \frac{\Gamma\left[1+\frac{2}{m_i}\left(\frac{N-2}{N-1}\right)\right] + \Gamma^2\left[1+\frac{N-2}{m_i(N-1)}\right]}{\Gamma^2\left[1+\frac{N-2}{m_i(N-1)}\right]} + \left\{\ln\ln 2 - m_i\left(\frac{N+1}{N-2}\right)\ln\Gamma\left[1+\frac{N-2}{m_i(N-1)}\right]\right\}^2.$$

For typical ceramic materials, $m_i\left(\frac{N+1}{N-2}\right) \geq 2$ and $K_d \simeq$

Since these derivations made a number of critical assumptions, it is of importance to compare these variances with those generated by the Monte Carlo technique to determine the validity of the assumptions.

2.2 Static Fatigue

Static fatigue tests entail the repeated measurement of failure time at several constant applied stresses. Static fatigue data can be analyzed similar to dynamic fatigue data by the median, homologous stress, iterative bivariant, and iterative trivariant analyses.[2,7] As in the case of dynamic fatigue the median analysis was chosen for studying the statistical reproducibility. With the median analysis the data is fitted to:[2,7]

$$\ln \hat{t}_f = a_3 \ln \sigma_a + a_4 \tag{10}$$

where $\hat{t}_f$ = median failure time and a_3, a_4 = linear regression constants. The fatigue constants N and B are determined from:[2,]

$$N = -a_3 \tag{11a}$$

$$\ln B = a_4 - (N-2) \ln \hat{\sigma}_1 \tag{11b}$$

The Monte Carlo analysis of the statistical reproducibility of N and B as determined from static fatigue data is similar to that used for dynamic fatigue data. First, a given number of failure times at a specific applied stress are selected by randomly choosing the failure probability and then calculating the corresponding failure time from:[2]

$$\ln t_f = \frac{(N-2)}{n_1} \ln \ln \frac{1}{1-F} - N \ln \sigma_a + (N-2) \ln \sigma_{01} - \ln B \tag{12}$$

Once a set of failure times at several different applied stresses and a set of inert strengths are randomly chosen, N and B are determined from Eqs. (10) and (11). By iterating this procedure 100 times, a distribution of N and B values are generated from which the statistical reproducibility of these parameters can be determined. Similar to the case of dynamic fatigue, the variables studied for a given material/environment system were the number of samples per applied stress, the applied stress range, and the number of applied stresses in a given test.

The variances in the fatigue constants N and B and their covariance could also be derived from statistical theory as shown in the Appendix, Eqs. (A52), (A53), and (A54), to be:

$$V(N) \simeq \frac{K_2 (N-2)^2}{J_0 m_1^2 R(\ln \sigma_a)} \tag{13a}$$

$$V(\ln B) = \frac{K_2 (N-2)^2}{J_0 m_1^2 R(\ln \sigma_a)} \left[(\ln \hat{S}_1 - \overline{\ln S_a})^2 + R(\ln \sigma_a) \right] + \frac{(N-2)^2 K_1}{J_1 m_1^2} \tag{13b}$$

$$Cov(N, \ln B) = \frac{K_2 (N-2)^2}{J_0 m_1^2 R(\ln \sigma_a)} \left[\overline{\ln S_a} - \ln \hat{S}_1 \right] \tag{13c}$$

where now J_0 = number of samples used in static fatigue experiment, J_2 = number of applied stresses, J_1 = number of samples tested at each applied stress,

$$R(\ln \sigma_a) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \sigma_{a_j} - \overline{\ln \sigma_a})^2, \quad \overline{\ln \sigma_a} = \frac{1}{J_2} \sum_{j=1}^{J_2} \ln \sigma_{a_j},$$

$$K_2 = \frac{1}{(N-2)^2} \frac{\Gamma\left[1 - \frac{2(N-2)}{m_1}\right] - \Gamma^2\left[1 + \frac{N-2}{m_1}\right]}{\Gamma^2\left[1 + \frac{(N-2)}{m_1}\right]} - \left[\ln \sigma_a - \frac{m_1}{N-2} \ln \Gamma\left(1 + \frac{N-2}{m_1}\right)\right]^2 \cdots$$

$$K_i = m_i^{\frac{1}{2}} \frac{\Gamma(1 - \frac{1}{m_i}) - \Gamma^2(1 + \frac{1}{m_i})}{\Gamma^2(1 - \frac{1}{m_i})} - [\text{illegible}]$$ For

$m_i > 2$, $K_i = 1.44$.

Again it was of interest to compare these variances to those determined by the Monte Carlo technique to determine the validity of the assumptions made in deriving the above equations.

3.0 RESULTS AND DISCUSSION

Both the Weibull parameters (m and S_o) and the fatigue constants (N and B) depend strongly on sample size. Figures 2 and 3 show the dependency for m and S_o, respectively on sample size where the coefficient of variation is defined as the standard deviation divided by the mean value of the parameter in question. From these figures it can be seen that there is good agreement between the Monte Carlo technique for estimating the statistical reproducibility and that derived from statistical theory. The figures further show that the statistical variability in the Weibull slope parameter m is a function of only sample size while, for the Weibull scale parameter S_o, the variability is a function of both sample size and the Weibull slope. Since for small sample sizes the statistical uncertainty in the Weibull parameters can be large, especially for m, important judgements and significant analyses of strength should not be based on small sample sizes. From Fig. 2 and 3 it can be seen that sample sizes of at least 30 should be used for all but the most preliminary investigations, although for small m's to get acceptable levels of S_o may require as many as 100 samples. Similar results and conclusions were reached in earlier Monte Carlo simulation studies.[12, 13]

Figures 4 and 5 show that the statistical reproducibility of the fatigue parameter N and B as determined by the dynamic fatigue experiment is strongly dependent on both sample size and the fatigue resistance of the material (large N values generally represent materials with a greater fatigue resistance). For sample sizes of less than 100, the statistical uncertainty in N and B can be very large, especially for the more fatigue resistant material. Again there is good agreement between the Monte Carlo technique for estimating the statistical reproducibility and that derived from statistical

theory; thus, giving evidence of the validity of the assumptions made in deriving the equations for statistical variability (see Appendix).

The statistical variability of the fatigue parameters N and B as determined by the dynamic fatigue experiment was also dependent on the number of stressing rates chosen, stressing rate range, and the Weibull slope m. For example, Fig. 6 shows that for the same range of stressing rates (maximum to minimum) and the same total number of samples, uncertainty in statistical reproducibility increased as the number of stressing rates used in determining N is increased from 2 to 7. The best statistical reproducibility occurs for the case where N is determined from strength measurements at two stressing rates corresponding to the maximum and minimum. Figure 7 shows the statistical variability of N as a function of m, keeping the other parameters constant, and illustrates that low m values, corresponding to a greater variability in strength, result in a larger uncertainty in N. Figure 8 shows that the statistical variability of N is quite sensitive to the stressing rate range, with the variability increasing with a decreasing range of stressing rates. Finally, it should be noted that the Monte Carlo results were left out of Figs. 6, 7, and 8 for clarity; however, these results agreed quite well with those shown in the figures based on statistical theory.

The distributions of the fatigue parameters N and B as generated by the Monte Carlo computer simulation technique for the dynamic fatigue experiment could be approximated by a normal distribution. Figure 9 shows that a normal distribution well represents the histogram for the fatigue parameter N. Histograms for the Weibull parameters m and S_o could also be approximated by normal distributions. This is important because the confidence limits of reproducibility for a given parameter can be estimated by simply multiplying

the standard deviation by the appropriate factor for normal distributions. Thus, the 95% reproducibility limits would be ±1.960 standard deviations[11] and represent the interval where about 95% of the values for a given parameter lie. The 90% reproducibility limits would be ±1.645 standard deviations[11] and so on.

Results on the statistical reproducibility of the fatigue parameters N and B as determined by the static fatigue experiment were similar to those shown for the dynamic fatigue experiment. In particular, the statistical variability of N and B as determined by the static fatigue experiment decreased with increasing sample size, decreased for materials with lower N values, decreased as the applied stress range increased, and decreased as the Weibull slope parameter m increased. For the same applied stress range and number of total samples, the best reproducibility occurs for the case where N and B are determined from time-to-failure measurements at two applied stresses corresponding to the maximum and minimum. For the case of multi-applied stresses, the statistical variability depended mainly on the total number of samples, not on the number of samples per applied stress, and only somewhat on the number of applied stresses used to determine N and B. As before, there was good agreement between the Monte Carlo analysis and the statistically derived variability equations (see Appendix). Also, the distributions generated by the Monte Carlo analysis could be approximated by normal distributions. Finally, it should be noted that the magnitude of the statistical reproducibility of N and B determined by the static fatigue experiment is similar to that determined from the dynamic fatigue experiment for typical ranges of dynamic and static fatigue data,[5] see Fig. 8. This could be expected since the statistical variability of the two experiments depends mainly upon the same four parameters: m, σ_o, N, and B.

Finally, it should be noted that there was a small bias present in the Monte Carlo generated values for m, S_o, N, and B, especially for small sample sizes. It is well known[11] that when random variables are combined in a non-linear fashion, the resulting quantity is generally subject to biases, i.e. the combined effect of the random fluctuations of the individual variables will cause the derived quantity to be systematically larger or smaller than it would have been in the total absence of such fluctuations. Biases in m and S_o have been previously discussed.[12,13,15] These systematic biases in the derived parameters were quite small in comparison with the statistical reproducibility of the parameters; hence, they were neglected in the statistical reproducibility of a given parameter.

4.0 PRACTICAL CONSIDERATIONS

Once the statistical reproducibility of the fatigue parameters N and B is known as a function of sample size and the other test variables, the corresponding statistical uncertainty in failure predictions can then be determined as discussed previously.[4,6] It is from this knowledge that the number of samples required for a given dynamic or static fatigue test can be determined. The exact choice of sample size for a given test will depend on the degree of reproducibility that is acceptable as well as the cost of testing; however, it always has to be remembered that erroneous judgements can be made and unacceptable designs pursued if sample sizes are too small. To illustrate how judgements regarding sample size can be made based on knowledge of statistical reproducibility, an example will be discussed.

Figure 9 shows a minimum lifetime prediction diagram for soda-lime glass in water based on Eq. (2). The limits shown on this figure represent the uncertainty due to the statistical variability in the fatigue constants N and B. These fatigue parameters were assumed to be determined by the dynamic fatigue test technique using three stressing rates with a stressing rate range of 32 to 0.013 MPa/s. The limits shown on the figure are $\pm$ 1.645 standard deviations, corresponding to the 90% confidence limits, and are a function of the number of samples used in the assumed dynamic fatigue test. To illustrate how this figure can be used, suppose that a certain glass component is to have a minimum lifetime of 10 years under an applied stress of 7 MPa. Based on the mean values in Fig. 9, the required proof stress is 32.4 MPa. Figure 10 is derived from Fig. 9 and shows the statistical uncertainty in the proof stress as a function of the number of samples used

in the dynamic fatigue test. An uncertainty of ± 10% requires about 140 samples to be used in the dynamic fatigue test for the stressing rate range of 0.013 to 50 MPa/s. However, it can be seen from the Fig. 12 that if the stressing rate range is increased to 0.0005 to 50 MPa/s, then the number of samples required would decrease to about 75 On the other hand, if the stressing rate range is decreased to 0.05 to 5.0 MPa/s, the number of samples required increases to about 300. This figure clearly shows that sample size requirements are not necessarily small and depend on the specific degree of reproducibility that is acceptable.

The analysis of statistical reproducibility can also be useful in determining whether a given variable effects the fatigue benavior of a material. For example, the fatigue behavior of soda-lime glass has been measured as a function of test environment (6N NaOH, distilled water, and 6N HCl) using the dynamic fatigue test with stressing rate range of .06 to 0.17 MPa/s and a total of 120 samples for each dynamic fatigue test.[16] The effect of test environment can be seen by calculating the allowable stress for samples that have been proof tested up to 100 MPa and that must survive a minimum of one year in service. The result of these calculations are given in Table I with the indicated ± one standard deviation limits due to the statistical uncertainty in the fatigue constants N and B. It is seen that the three predicted allowable stresses are well outside of a standard deviation from each other and, thus, it is likely that the test environments do have a significant effect on the fatigue behavior of soda-lime glass. While this conclusion seems quite straightforward, it must be remembered that before any meaningful conclusions can be drawn regarding the effect of a particular variable on the fatigue behavior of a material, it must be demonstrated that the effect is larger than the statistical reproducibility of the experiment.

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Table 1. Predicted Allowable Applied Stress for a Minimum Lifetime of One-Year for Soda Lime Glass Samples Proof Tested at 100 MPa

Environment	N	ln B ($MPa^2 \cdot s$)	σ_a (MPa)
6N NaOH	19.5 ($\pm$ 2.81)+	0.168 ($\pm$ 1.64)	25.95 ($\pm$ 3.36)
Distilled H_2O	13.0 ($\pm$ 1.11)	2.585 ($\pm$ 0.87)	15.92 ($\pm$ 1.58)
6N HCl	25.1 ($\pm$ 4.35)	-16.058 ($\pm$ 4.67)	18.36 ($\pm$ 2.15)

+ Number in parenthesis is $\pm$ one standard deviation corresponding to statistical reproducibility of dynamic fatigue experiment.

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FIGURES

Figure 1. Flow diagram for Monte Carlo technique for evaluating the statistical reproducibility of N and B as determined from the dynamic fatigue experiment.

Figure 2. The coefficient of variation (C.V.) of Weibull slope parameter m as a function of sample size for various values of m. The Monte Carlo results are given by the data points and are compared to predictions from statistical theory, Eq.(8a).

Figure 3. The coefficient of variation of Weibull scale parameter S_o for various values of m as a function of sample size. The Monte Carlo results are given by the data points and are compared to predictions from statistical theory, Eq.(8b).

Figure 4. The coefficient of variation of fatigue parameter N as a function of total number of samples used in dynamic fatigue experiment with 3 stressing rates (0.013 to 50 MPa/s). The Monte Carlo results are given by the data points and are compared to predictions from statistical theory, Eq.(9a). The Weibull and fatigue constants chosen are typical of soda-lime glass[2] (m_i = 15.0, S_{oi} = 138.0 MPa, N = 18.4, and B = 0.18 $MPa^2 \cdot s$) and vitrified grinding wheel material[14] (m_i = 15, S_{oi} = 53 MPa, N = 43.2 and B = 1.1 x 10^{-5} $MPa^2 \cdot s$).

Figure 5. The coefficient of variation of fatigue parameter B as a function of total number of samples used in dynamic fatigue experiment with 3 stressing rates (0.013 to 50 MPa/s). The Monte Carlo results are given by the data points and compared to predictions from statistical theory, Eq.(9b). The Weibull and fatigue constants chosen are typical of soda-lime glass[2] (m_i = 15.0, S_{oi} = 138.0 MPa, N = 18.4, and B = 0.18 $MPa^2 \cdot s$) and vitrified grinding wheel material[14] (m_i = 15, S_{oi} = 53 MPa, N = 43.2 and B = 1.1 x 10^{-5} $MPa^2 \cdot s$).

Figure 6. The coefficient of variation of fatigue parameter N as a function of total number of samples used in dynamic fatigue experiment with stressing rate range of 0.013 to 50 MPa/s. The number of stressing rates used in determining N is as indicated and the appropriate constants are N = 18.4, B = 0.18 $MPa^2 \cdot s$, m_i = 8.2, and S_{oi} = 138.0 MPa.

Figure 7. The coefficient of variation of fatigue parameter N as a function of total number of samples used in dynamic fatigue experiment for 3 stressing rates (0.013 to 50 MPa/s) and as a function of Weibull slope parameter m_i, while the other constants are N = 18.4, B = 0.18 $MPa^2 \cdot s$, and S_{oi} = 138.0 MPa).

Figure 8. The coefficient of variation of fatigue parameter N as a function of total number of samples used in dynamic fatigue experiment for 3 stressing rates with two ranges: 0.013 to 50 and 0.05 to 5.0 MPa/s. The appropriate constants are N = 18.4, B = 0.18 $MPa^2 \cdot s$, m_i = 8.2, and S_{oi} = 138.0 MPa.

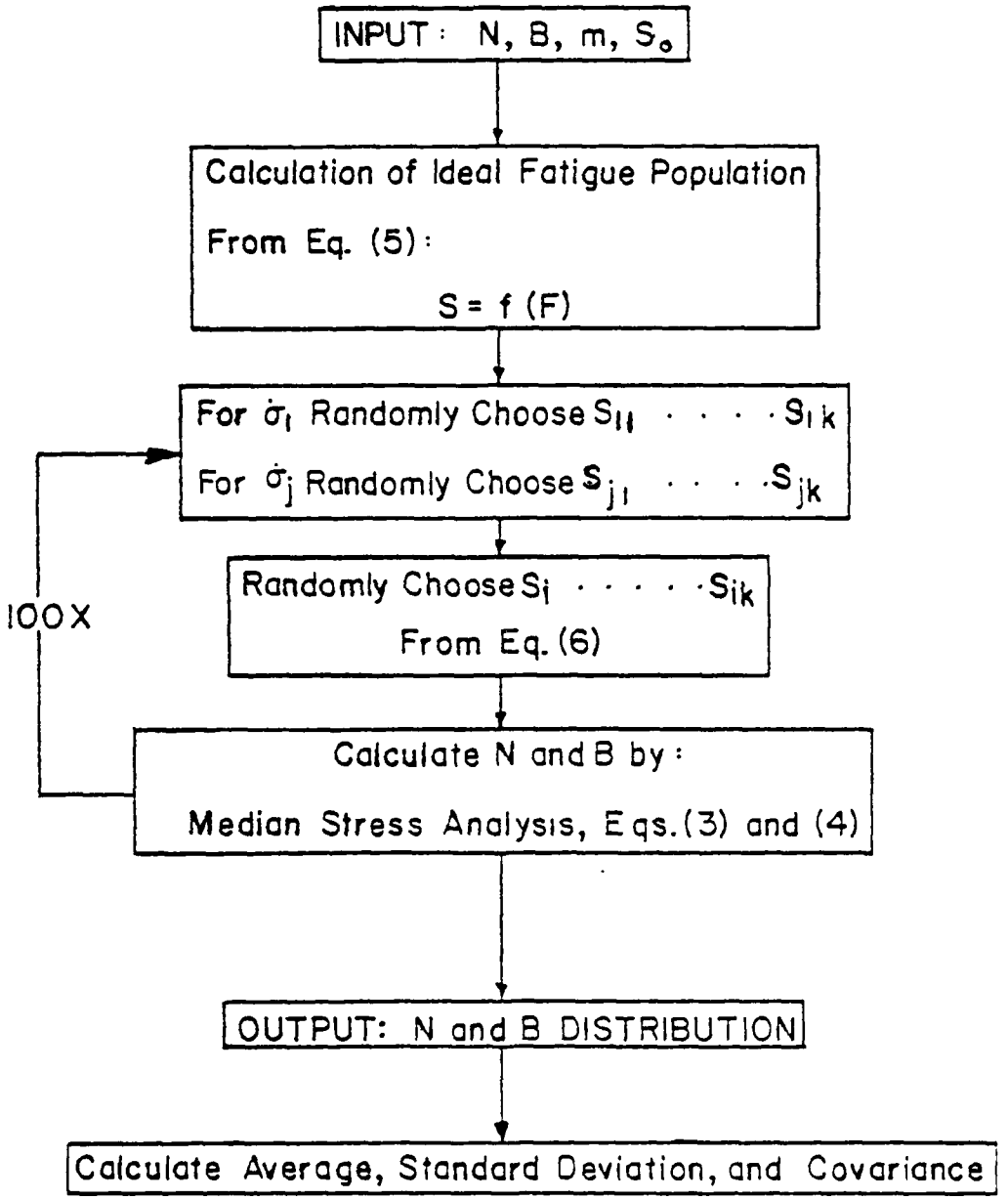
Figure 9. Histogram for fatigue parameter N as determined Monte Carlo simulated dynamic fatigue experiment for 3 stressing rates (0.013 to 50 MPa/s) using a total of 150 samples. The superimposed smooth curve is the probability density function of the normal distribution with the same mean and variance as the Monte Carlo results.

Figure 10. The coefficient of fatigue parameter N as a function of total number of samples used in dynamic fatigue experiment with 3 stressing rates (0.013 to 50 MPa/s) and static fatigue experiment with 3 applied stresses (40 to 58.6 MPa). Experiments are typical for soda-lime glass[2] (N = 18.4, B = 0.18 $MPa^2 \cdot s$, m_i = 8.2, and S_{oi} = 138.0 MPa).

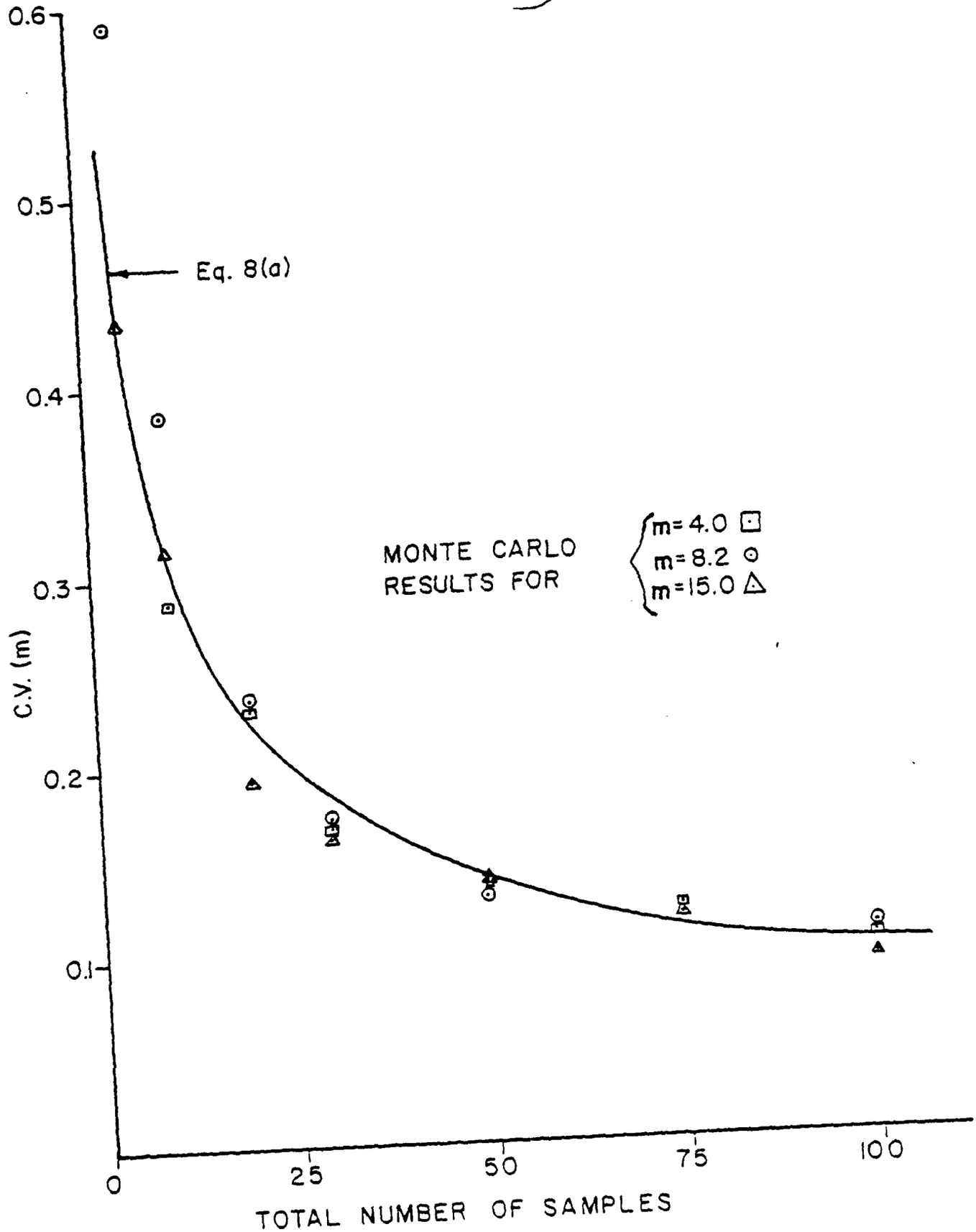
Figure 11. Design diagram for minimum lifetime prediction after proof testing for soda lime glass (N = 18.4, B = 0.18 $MPa^2 \cdot s$, m_i = 8.2, and S_{oi} = 138.0 MPa) The 90% confidence limits are shown for 3 sample sizes (45, 90, and 180) used in dynamic fatigue experiment with 3 stressing rates (0.013 to 50 MPa/s).

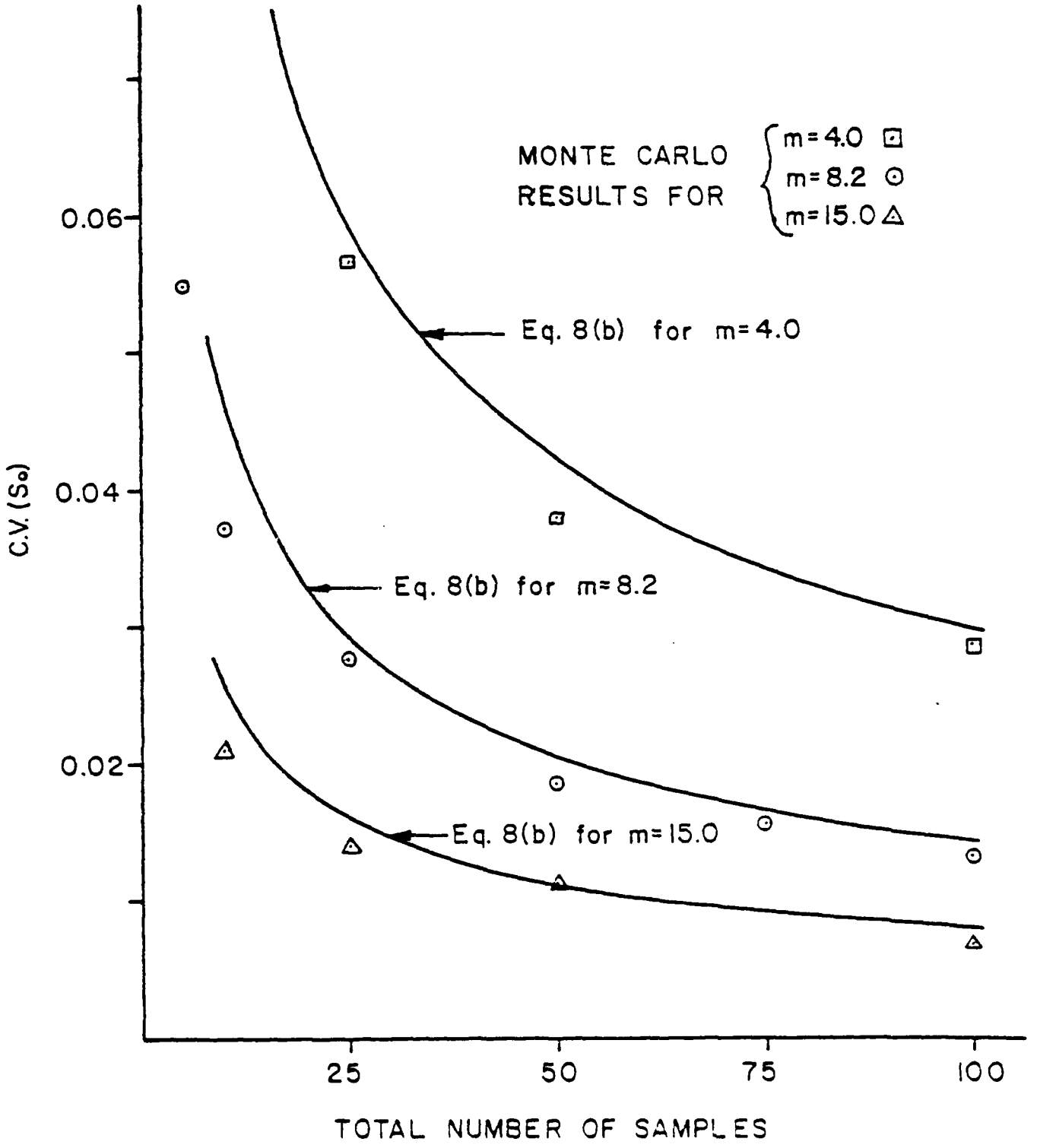
Figure 12. 90% confidence limits for the proof stress to assure a minimum lifetime of 10 years at a constant applied stress of 7 MPa for soda-lime glass (N = 18.4, B = 0.18 $MPa^2 \cdot s$, m_i = 8.2, S_{oi} = 138.0 MPa) as a function of sample size used in dynamic fatigue experiment with 3 stressing rates in the range of 0.05 to 5.0, 0.013 to 50, and 0.005 to 50 MPa/s.

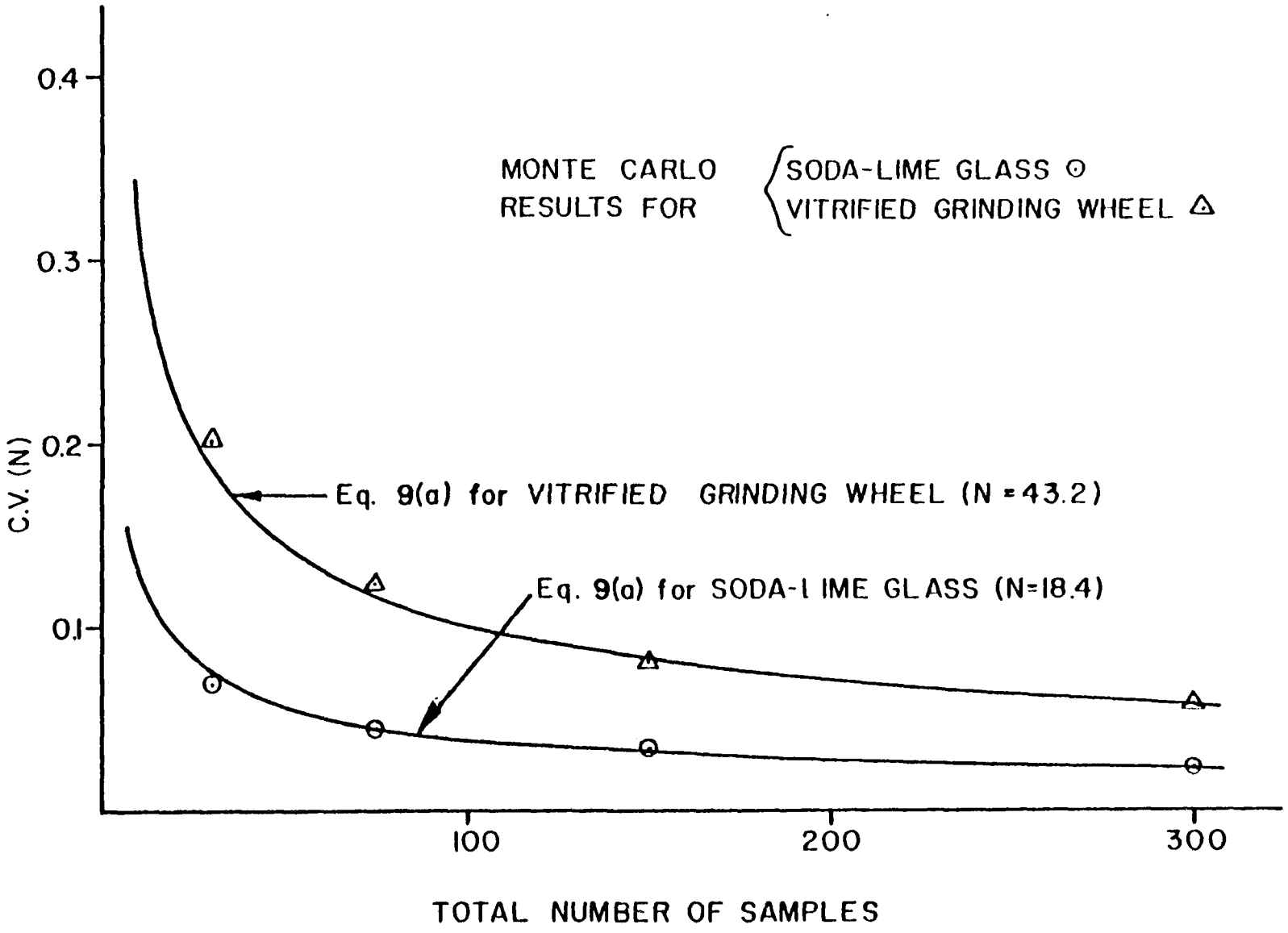
(1)

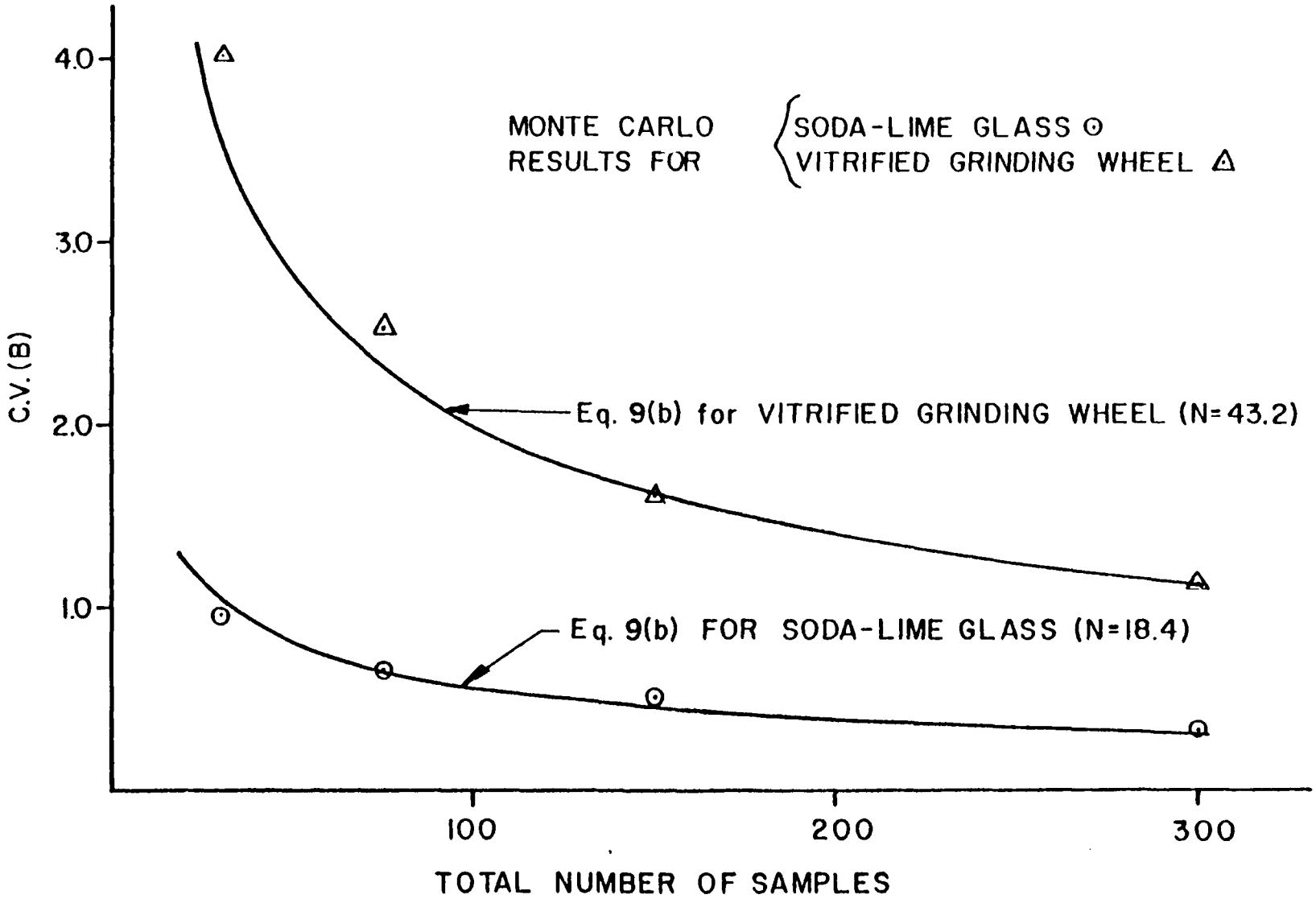

INPUT: N, B, m, S_o
Calculation of Ideal Fatigue Population
From Eq. (5):
$S = f(F)$
For $\dot{\sigma}_i$ Randomly Choose $S_{i1} \cdots S_{ik}$
For $\dot{\sigma}_j$ Randomly Choose $S_{j1} \cdots S_{jk}$
Randomly Choose $S_i \cdots S_{ik}$
From Eq. (6)
Calculate N and B by:
Median Stress Analysis, Eqs. (3) and (4)
100X
OUTPUT: N and B DISTRIBUTION
Calculate Average, Standard Deviation, and Covariance

2

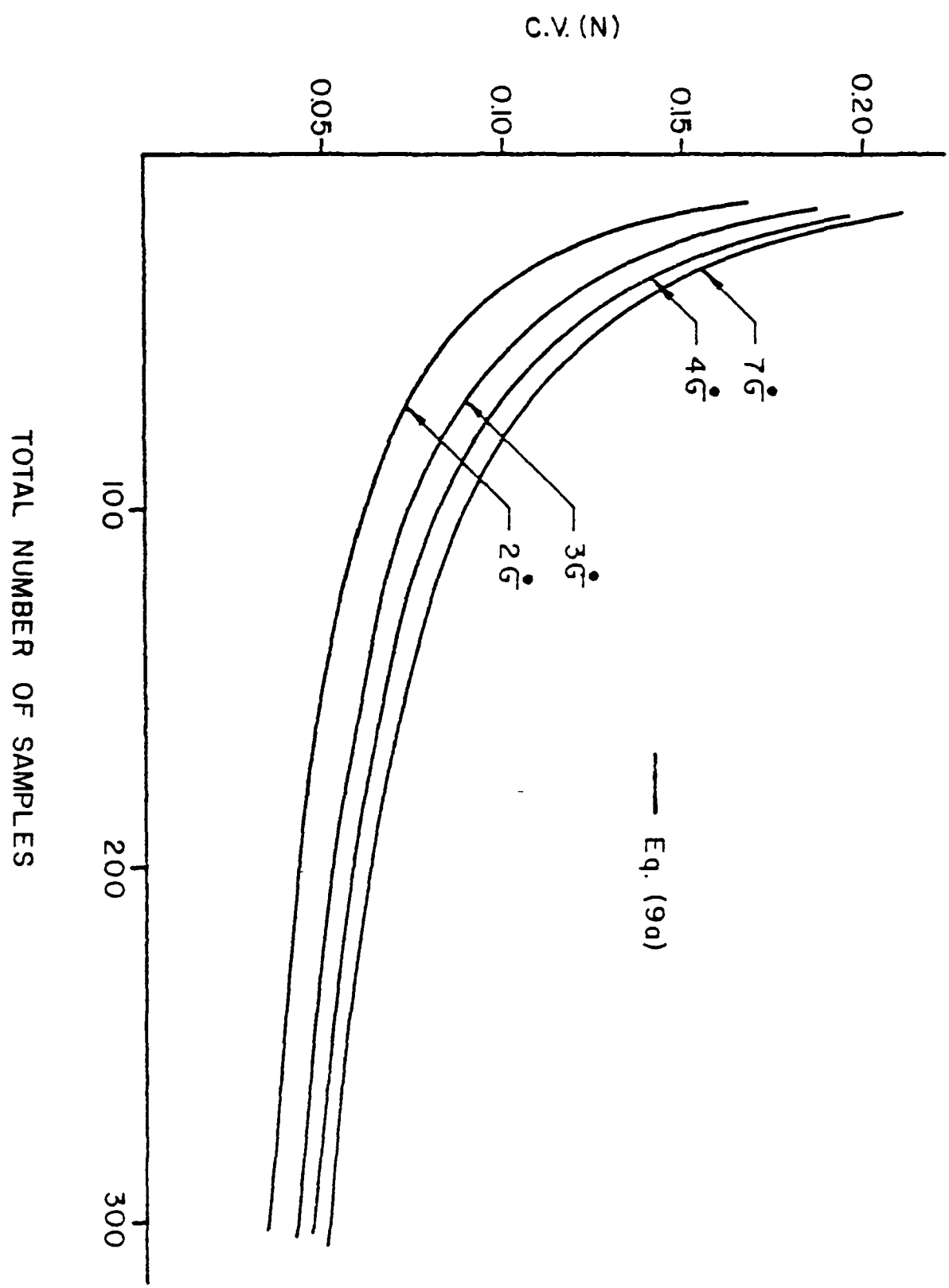

0.6
0.5
0.4
0.3
0.2
0.1
C.V. (m)
Eq. 8(a)
MONTE CARLO RESULTS FOR
m=4.0
m=8.2
m=15.0
0
25
50
75
100
TOTAL NUMBER OF SAMPLES

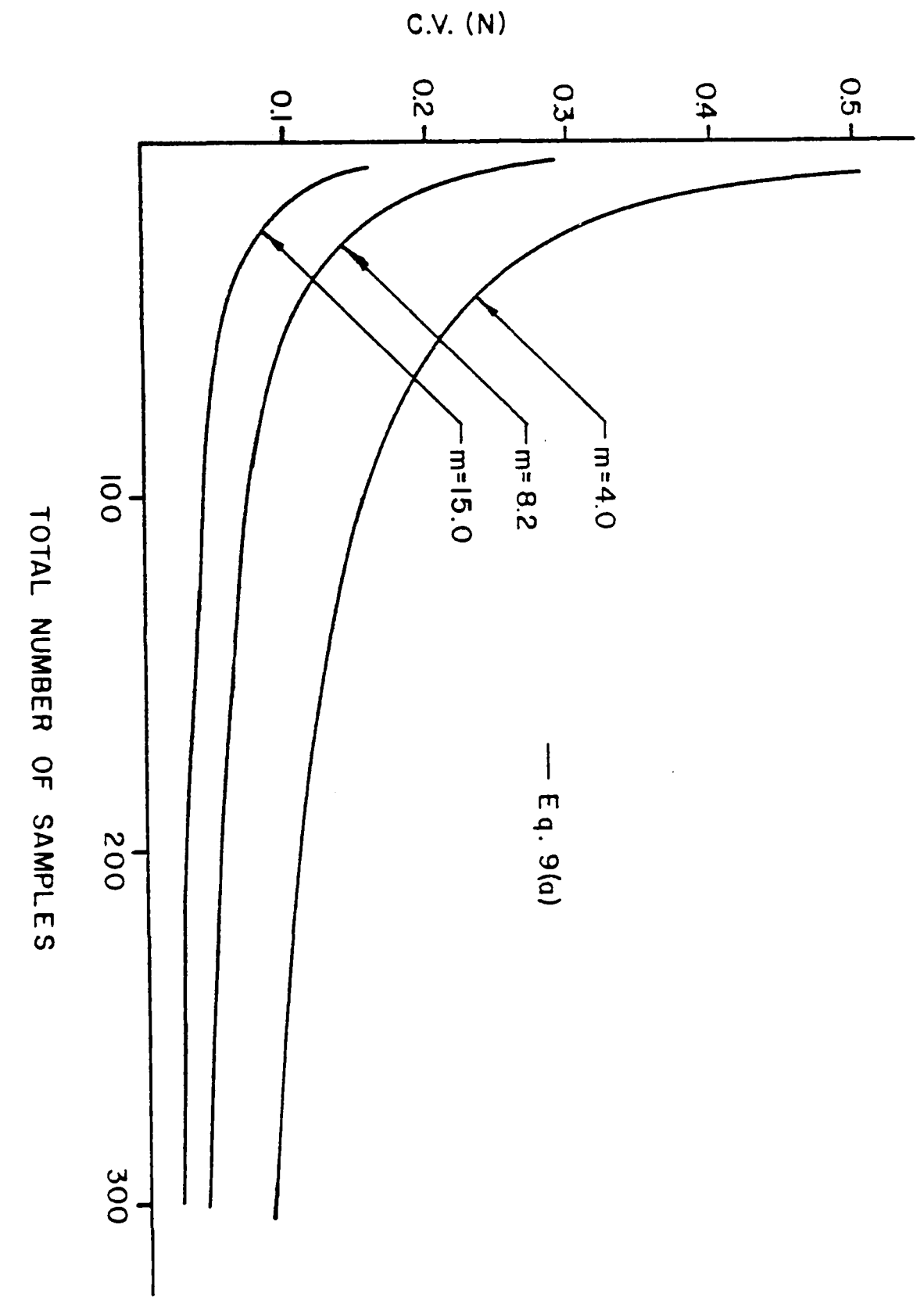

MONTE CARLO RESULTS FOR
m=4.0
m=8.2
m=15.0
Eq. 8(b) for m=4.0
Eq. 8(b) for m=8.2
Eq. 8(b) for m=15.0
0.06
0.04
0.02
C.V. (S₀)
25
50
75
100
TOTAL NUMBER OF SAMPLES

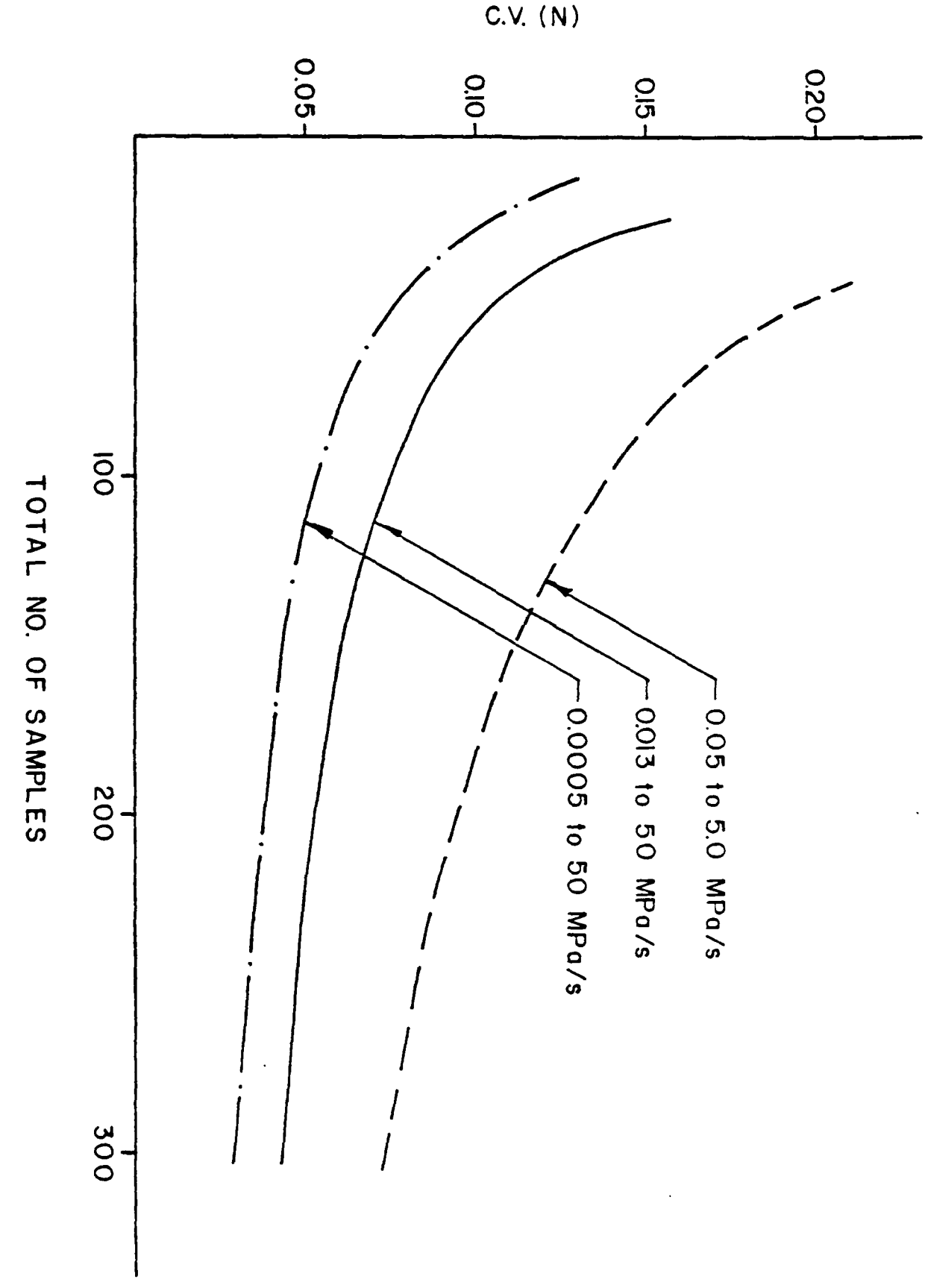

0.4
0.3
0.2
0.1
C.V. (N)
MONTE CARLO RESULTS FOR
SODA-LIME GLASS ⊙
VITRIFIED GRINDING WHEEL △
Eq. 9(a) for VITRIFIED GRINDING WHEEL (N = 43.2)
Eq. 9(a) for SODA-LIME GLASS (N=18.4)
100
200
300
TOTAL NUMBER OF SAMPLES


MONTE CARLO RESULTS FOR
SODA-LIME GLASS ⊙
VITRIFIED GRINDING WHEEL △
Eq. 9(b) for VITRIFIED GRINDING WHEEL (N=43.2)
Eq. 9(b) FOR SODA-LIME GLASS (N=18.4)
C.V. (B)
4.0
3.0
2.0
1.0
100
200
300
TOTAL NUMBER OF SAMPLES

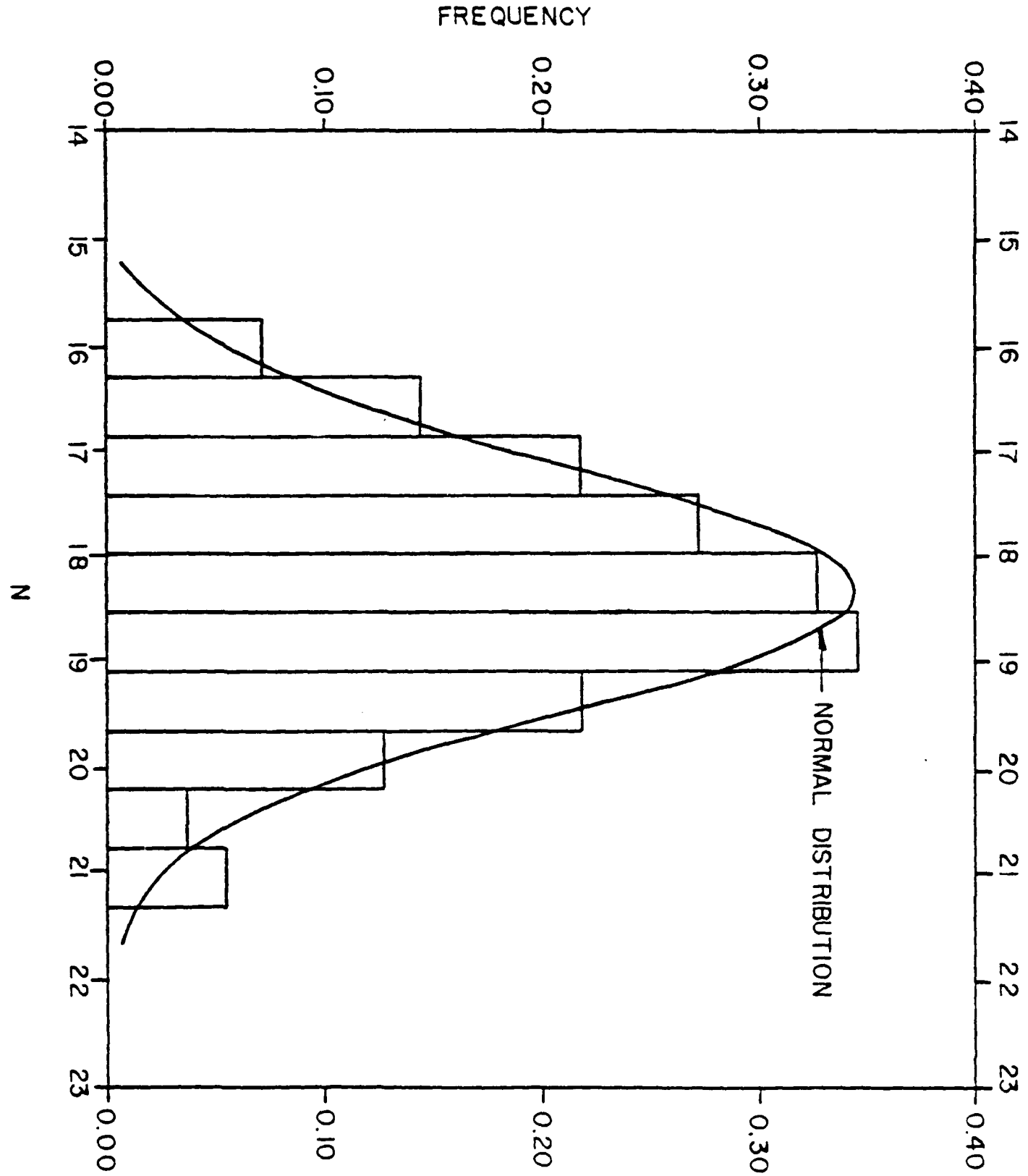
5

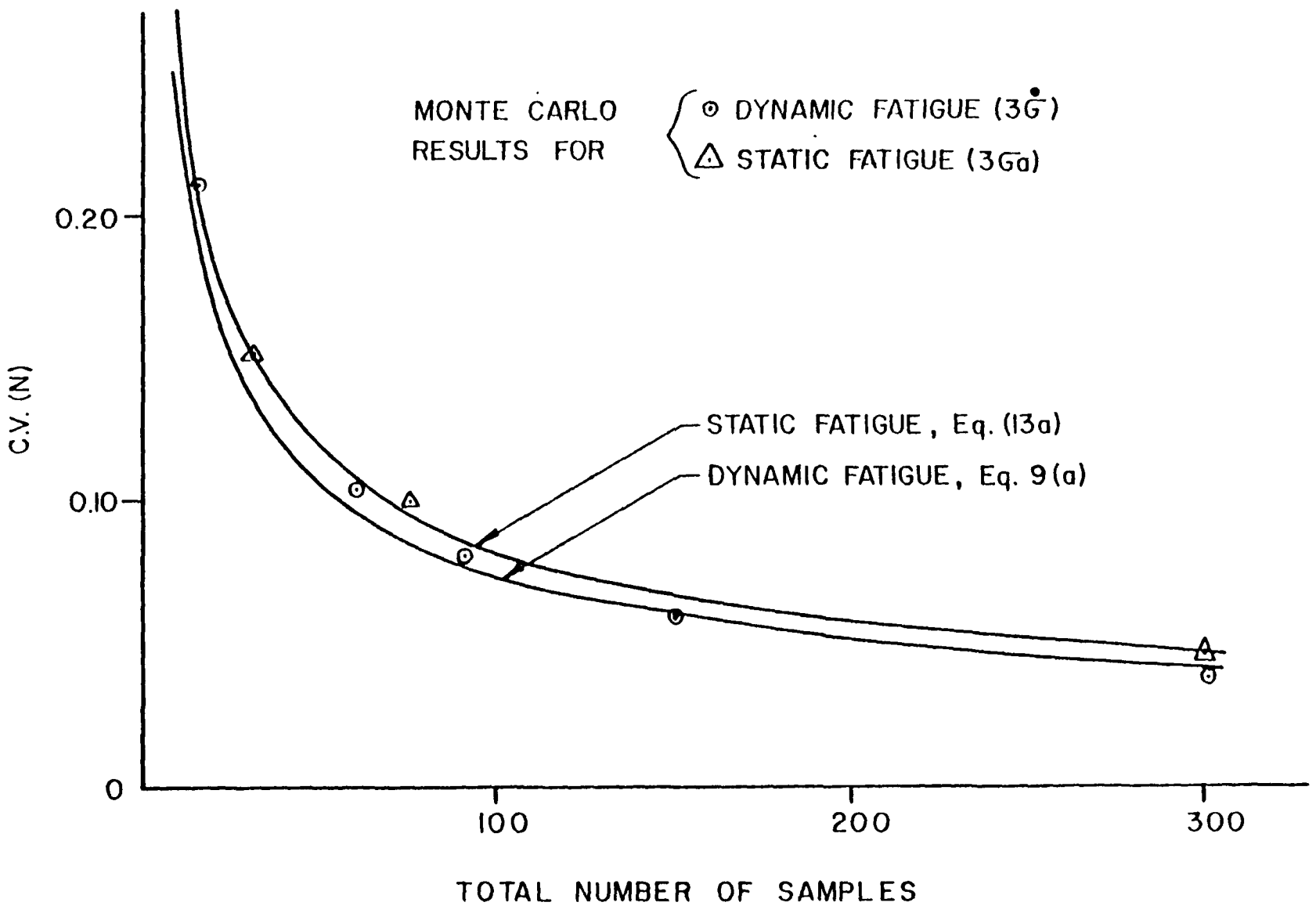

C.V. (N)
0.05
0.10
0.15
0.20
100
200
300
TOTAL NUMBER OF SAMPLES
$7\bar{G}^{\bullet}$
$4\bar{G}^{\bullet}$
$3\bar{G}^{\bullet}$
$2\bar{G}^{\bullet}$
— Eq. (9a)

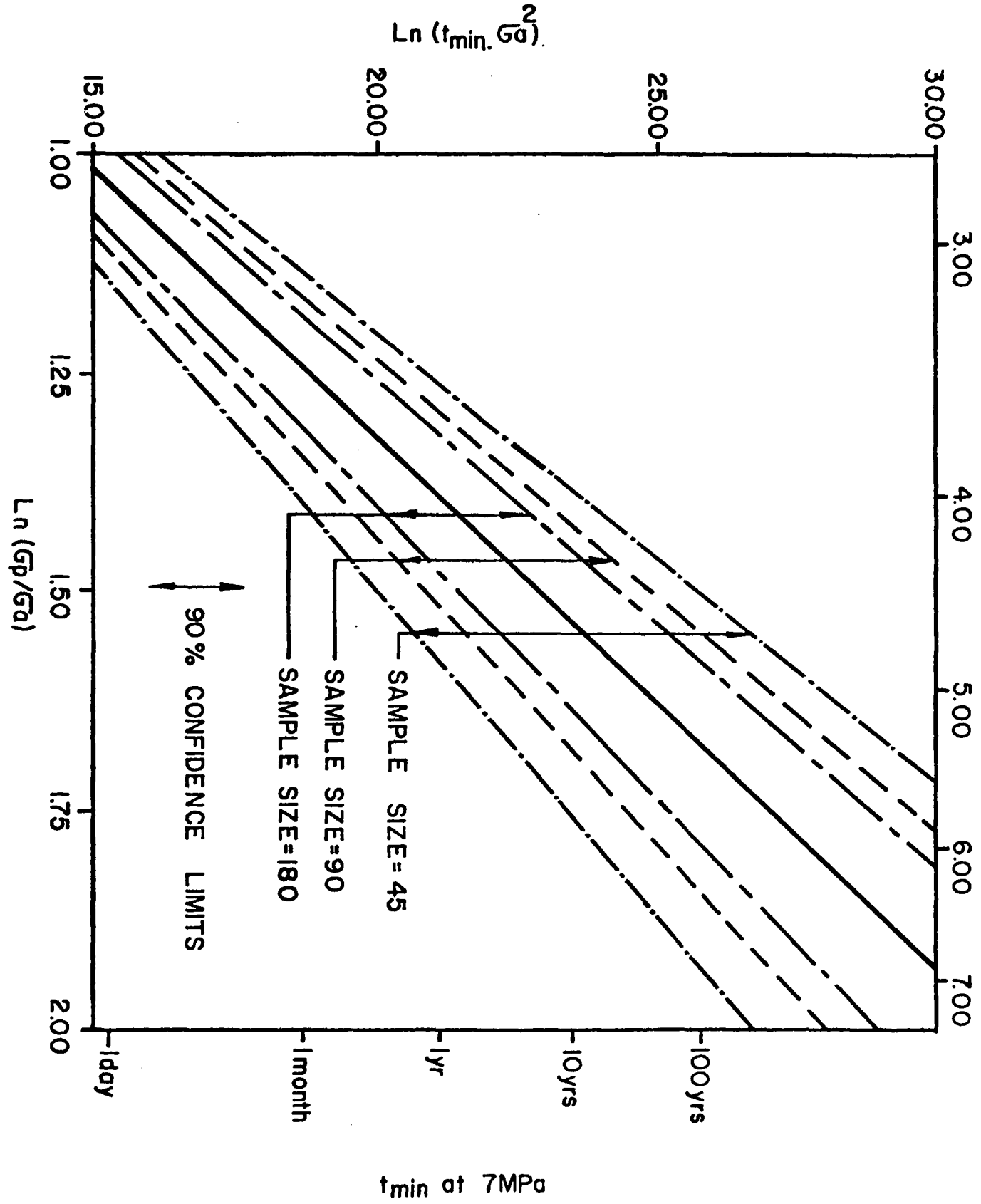

C.V. (N)
0.1
0.2
0.3
0.4
0.5
m=4.0
m=8.2
m=15.0
— Eq. 9(a)
100
200
300
TOTAL NUMBER OF SAMPLES

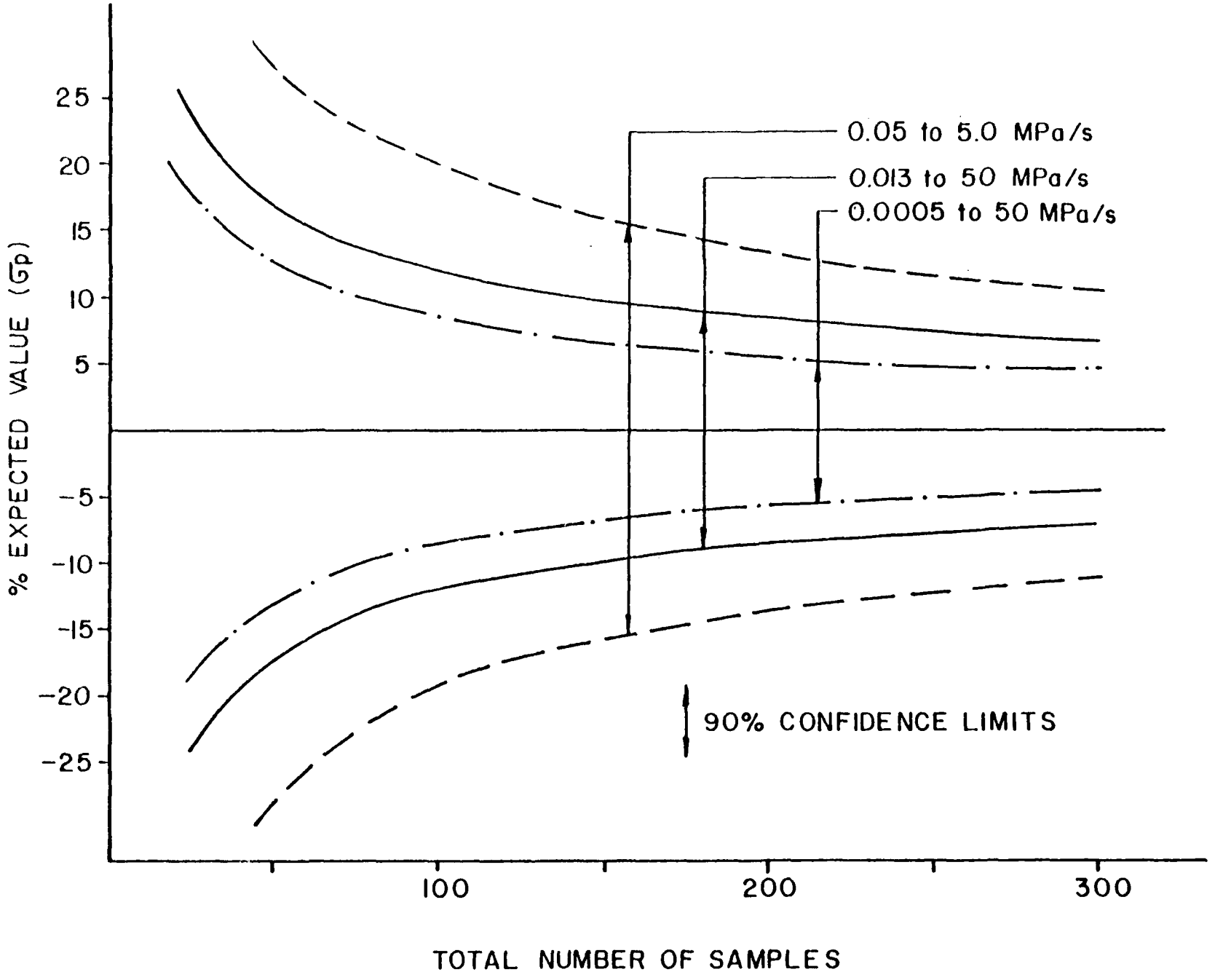

C.V. (N)
0.05
0.10
0.15
0.20
100
200
300
TOTAL NO. OF SAMPLES
0.05 to 5.0 MPa/s
0.013 to 50 MPa/s
0.0005 to 50 MPa/s

(9)


FREQUENCY
0.00
0.10
0.20
0.30
0.40
14
15
16
17
18
19
20
21
22
23
N
NORMAL DISTRIBUTION


MONTE CARLO RESULTS FOR
⊙ DYNAMIC FATIGUE ($3\dot{G}$)
△ STATIC FATIGUE ($3G_a$)
0.20
0.10
0
C.V. (N)
STATIC FATIGUE, Eq. (13a)
DYNAMIC FATIGUE, Eq. 9(a)
100
200
300
TOTAL NUMBER OF SAMPLES


Ln (t_{min}. $\bar{G}a^2$)
15.00
20.00
25.00
30.00
1.00
1.25
1.50
1.75
2.00
Ln ($\bar{G}p/\bar{G}a$)
3.00
4.00
5.00
6.00
7.00
($\bar{G}p/\bar{G}a$)
90% CONFIDENCE LIMITS
SAMPLE SIZE=180
SAMPLE SIZE=90
SAMPLE SIZE=45
1day
1month
1yr
10yrs
100yrs
t_{min} at 7MPa


25
20
15
10
5
-5
-10
-15
-20
-25
% EXPECTED VALUE (Gp)
0.05 to 5.0 MPa/s
0.013 to 50 MPa/s
0.0005 to 50 MPa/s
90% CONFIDENCE LIMITS
100
200
300
TOTAL NUMBER OF SAMPLES

APPENDIX

A.1. Linear Regression

Linear regression is generally used to analyze the test data in determining the fatigue parameters N and B. The variances of these fatigue parameters can then be estimated by using the law of propagation of errors, commonly known as the chain rule.[11,17] The chain rule is a linear approximation and becomes inaccurate when the coefficient of variation (ratio of the standard deviation to the mean) of the data is large. Mandel[11] gives the rule of thumb that the coefficient of variation should not exceed 10%.

In linear regression J data values are least squares fitted to a straight line whose equation is given by:

$$y = ax + b \tag{A1}$$

where y is the dependent variable, x is the independent variable, a is the slope, and b is the intercept. Assuming that all the inaccuracies occur in the measured values of y and that the parent standard deviation of the data, σ, is independent of x, the variances and covariance of the parameters a and b can be expressed as:[17]

$$V(a) = \frac{\sigma^2}{\Delta(x)} J \tag{A2}$$

$$V(b) = \frac{\sigma^2}{\Delta(x)} \sum x_i^2 \tag{A3}$$

$$Cov(a, b) = -\frac{\sigma^2}{\Delta(x)} \sum x_i \tag{A4}$$

where $\Delta(x) = J \sum x_i^2 - (\sum x_i)^2$. By rearranging the equation for $\Delta(x)$:

$$\Delta(x) = \sigma^2 \left[\frac{\sum x_i^2 - \bar{x}^2}{\sigma^2} J^2 \right] = J^2 R(x) \tag{A5}$$

where $R(x_i) = \Sigma (x_i - \bar{x})^2 / n$ and $\bar{x} = \Sigma x_i / n$. Thus,

$$\Sigma x_i^2 \approx n [R(x_i) + \bar{x}^2] \quad (A6)$$

By using Eqs. (A5) and (A6), Eqs. (A3) - (A4) can now be expressed as:

$$V(a) = \frac{\sigma^2}{n R(x)} \quad (A7)$$

$$V(b) = \frac{\sigma^2}{n R(x)} [R(x) - \bar{x}^2] \quad (A8)$$

$$Cov(a, b) = \frac{-\sigma^2}{n R(x)} \bar{x} \quad (A9)$$

A.2. Weibull Parameters

The Weibull shape and scale parameters, m and S_o are typically determined by fitting the strength data to:[10]

$$\ln S = \frac{1}{m} Y_F - \ln S_o \quad (A10)$$

where $Y_F = \ln \ln \frac{1}{1-F}$, F is the cumulative failure probability, and S is the strength. Since Eq. (A10) is similar in form to Eq. (A1), the variances and covariance of the parameters 1/m and $\ln S_o$ can be determined from Eqs. (A7) - (A9).

Recognizing that in this case $\sigma^2 = V(\ln S)$ and that for sample sizes [illegible], $J \approx J - 1$, so that $R(X_F)$ can be taken to be $V(X_F) = [illegible] (X_F - \bar{X}_F)^2 [illegible]$, Eqs. (A7 - A9) can be rewritten:

$$V(1/m) = \frac{V(\ln S)}{J\,V(X_F)} \tag{A11}$$

$$V(\ln S_0) = \frac{V(\ln S)}{J\,V(X_F)}\left[V(X_F) + \bar{X}_F^{\,2}\right] \tag{A12}$$

$$\mathrm{Cov}(1/m, \ln S_0) = -\frac{V(\ln S)}{J\,V(X_F)}\,\bar{X}_F \tag{A13}$$

By applying the chain rule to Eq. (A10), one gets

$$V(\ln S) = \left(\frac{\partial \ln S}{\partial X_F}\right)^2 V(X_F) = \frac{1}{m^2}\,V(X_F) \tag{A14}$$

which on substituting into Eq. (A11) gives:

$$V(1/m) = \frac{1}{J\,m^2} \tag{A15}$$

From the chain rule,

$$V(m) = m^4\,V(1/m) = m^2/J \tag{A16}$$

The variance of $\ln S_0$ is obtained from Eq. (A12) and (A14) as:

$$V(\ln S_0) = \frac{1}{J}\left[V(\ln S) + \bar{X}_F^{\,2}/m^2\right] \tag{A17}$$

From Eq. (A10),

$$\bar{X}_F = \frac{1}{J}\sum m(\ln S - \ln S_0) = m(\overline{\ln S} - \ln S_0) \tag{A18}$$

so that,

$$V(\ln S_0) = \frac{1}{J}\left[V(\ln S) + (\overline{\ln S} - \ln S_0)^2\right] \tag{A19}$$

The covariance between 1/m and $\ln S_o$ is obtained from Eqs. A13, A14, and A18 to be:

$$Cov\left(1/m, \ln S_o\right) = \frac{\left(\ln S_o - \overline{\ln S}\right)}{-m} \qquad (A20)$$

Using the chain rule it can be shown that:

$$Cov\left(m, \ln S_o\right) = \left(\frac{dm}{d\,1/m}\right) Cov\left(1/m, \ln S_o\right) = m\frac{\left(\overline{\ln S} - \ln S_o\right)}{-1} \qquad (A21)$$

Equations (A16), (A19), and (A21) give the variances and covariances of the Weibull parameters m and $\ln S_o$ and can be analytically evaluated for any given m and S by noting that for a Weibull distribution the mean and variance of S are given by:[9, 15, 18]

$$\bar{S} = S_o\left[\Gamma\left(1 + 1/m\right)\right] \qquad (A22)$$

$$V(S) = S_o^2\left[\Gamma\left(1 + 2/m\right) - \Gamma^2\left(1 + 1/m\right)\right] \qquad (A23)$$

where Γ = gamma function. Since for typical ceramic strength data $\ln \bar{S} = \overline{\ln S}$, Eq. (A22) can be rewritten:

$$\overline{\ln S} = \ln S_o + \ln \Gamma\left(1 + 1/m\right) \qquad (A24)$$

From the chain rule,

$$V\left(\ln S\right) = \frac{V(S)}{\bar{S}^2} = \frac{\Gamma\left(1 + 2/m\right) - \Gamma^2\left(1 + 1/m\right)}{\Gamma^2\left(1 + 1/m\right)} \qquad (A25)$$

Substituting Eqs. (A24) and (A25) into Eqs. (A19) and (A21) yields,

$$V\left(\ln S_o\right) = \frac{1}{n}\left\{\frac{\Gamma\left(1 + 2/m\right) - \Gamma^2\left(1 + 1/m\right)}{\Gamma^2\left(1 + 1/m\right)} + \left[\ln \Gamma\left(1 + 1/m\right)\right]^2\right\} \qquad (A26)$$

$$Cov\left(m, \ln S_o\right) = \frac{m}{n} \ln \Gamma\left(1 + 1/m\right) \qquad (A27)$$

From the variances and covariance of the Weibull parameters, given in Eqs. (A16), (A26), and (A27), the variance of the median strength $\hat{S}_{.5}$ can be evaluated by substituting F = 0.5 into Eq. (A10) and using the chain rule to give:

$$V(\ln \hat{S}_{.5}) = \frac{\ln^2 2}{m^4} V(m) + V(\ln \hat{S}_0) - \frac{2 \ln 2}{m^2} \mathrm{Cov}(m, \ln \hat{S}_0) \qquad (A28)$$

Substituting Eqs. (A16), (A26), and (A27) into Eq. (A28), gives:

$$V(\ln \hat{S}_{.5}) = \frac{K}{n m^2} \qquad (A29)$$

where

$$K = m^2 \left[\Gamma(1+2/m) - \Gamma^2(1+1/m)\right] + \left[\ln 2 \ln 2 - m \ln \Gamma(1+1/m)\right]^2$$

For $m > 2$, $K = 1.44$ so that:

$$V(\ln \hat{S}_{.5}) \simeq \frac{1.44}{n m^2} \qquad (A30)$$

A most useful relationship can be derived from Eq. (A30) by recognizing that $V(\ln \hat{S}_{.5}) = V(\ln \bar{S})$ which in turn is equal to $V(\bar{S})/\bar{S}^2$ by the chain rule. Making these substitutions into Eq. (A30), one can derive that:

$$m \simeq \frac{1.2\,\bar{S}}{\sqrt{n}\left[V(\bar{S})\right]^{1/2}} = \frac{1.2\,\bar{S}}{\left[V(S)\right]^{1/2}} \qquad (A31)$$

This relationship shows that the Weibull slope can be estimated by simply dividing 1.2 by the coefficient of variation.

A.3. Dynamic Fatigue

By fitting dynamic fatigue data to Eq. (3), the variances and covariance of the regression coefficients a_1 and a_2 can be obtained by Eqs. (A27) - (A29) and the variances in N and ln B can then be derived using the chain rule. Assume that the dynamic fatigue data involves J_2 different stressing rates and that at each stressing rate J_1 number of fracture strengths are measured. The strength data at each stressing rate is fitted to the Weibull equation, Eq. (A10), to obtain the median strength values and their variances. For the j-th stressing rate, Eq. (A10) and (A29) can be written:

$$\ln \hat{\sigma}_{aj} = \frac{-1}{m_{dj}} \ln \ln 2 - \ln \sigma_{0dj} \tag{A32}$$

$$V(\ln \hat{\sigma}_{aj}) = \frac{K_{dj}}{J_1 m_{dj}^2} \tag{A33}$$

Equations (A32) and (A33) can be written in terms of the inert strength Weibull slope, m_i, since from Eq. (5):

$$m_{dj} = m_i \left(\frac{N+1}{N-2} \right) \tag{A34}$$

Since m_{dj} is independent of stressing rate, K_{dj} is also independent of stressing rate, hence, the subscript "j" can be dropped and:

$$K_{dj} = \sigma_{dj}^2 = m_i^2 \left(\frac{N+1}{N-2} \right)^2 \frac{\quad}{\quad} \tag{A35}$$

For $m_i \frac{(N+1)}{(N-2)} \geq 2$, $K_d \simeq 1.44$ to a good approximation.

Substituting Eqs. (A32) and (A33) into (A30) and (A31),

$$\ln \hat{\sigma}_{dj} = \frac{N-2}{m_i (N+1)} \ln \ln 2 + \ln \sigma_{dj} \tag{A36}$$

$$V(\ln \hat{\sigma}_{dj}) = \frac{K_d (N-2)^2}{J_i (N+1)^2 m_i^2} \tag{A37}$$

Equation (A37) defines the parent sample variance, σ^2, for the median fracture strength, hence the variances and covariance of a_1 and a_2 in Eq. (3) can be calculated from Eqs. (A7)-(A9) as:

$$V(a_1) = \frac{K_d (N-2)^2}{J_0 (N+1)^2 m_i^2 R(\ln \dot{\sigma})} \tag{A38}$$

$$V(a_2) = \frac{K_d (N-2)^2}{J_0 (N+1)^2 m_i^2} \left[1 + \frac{(\overline{\ln \dot{\sigma}})^2}{R(\ln \dot{\sigma})} \right] \tag{A39}$$

$$Cov(a_1, a_2) = - \frac{K_d (N-2)^2}{J_0 (N+1)^2 m_i^2} \frac{\overline{\ln \dot{\sigma}}}{R(\ln \dot{\sigma})} \tag{A40}$$

where $R(\ln \dot{\sigma}) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \dot{\sigma} - \overline{\ln \dot{\sigma}})^2$ and $J_0 = J_1 J_2$ = total number of samples used in the dynamic fatigue experiment.

The variances and covariance of N and ln B can be calculated by applying the chain rule to Eqs (4a) and 4b, and then substituting in Eqs. (A38) - (A40). Thus, the variance of N is:

$$V(a) = \frac{V(a)}{a_1^2} = \frac{K_d (N-2)^2 (N+1)^2}{J_0 m_i^2 R(\ln \dot{\sigma})} \tag{A41}$$

The variance of ln B is:

$$V(\ln B) = \frac{\partial \ln B}{\partial a_1}\Big|^2 V(a_1) + \frac{\partial \ln B}{\partial a_2}\Big|^2 V(a_2) + 2\frac{\partial \ln B}{\partial a_1}\frac{\partial \ln B}{\partial a_2}\Big|_{\ldots} \ldots$$

$$+ \frac{\partial \ln B}{\partial \ln \hat{S}_i}\Big|^2 V(\ln \hat{S}_i)$$

$$V(\ln B) = \frac{K_d (N-2)^2}{J_0 m^2 R(\ln \hat{S}_i)} \left\{ \left[3 \ln \hat{S}_i - 1 - \ln 3(N-1) - \overline{\ln \hat{S}} \right]^2 + R(\ln \hat{S}_i) - \ldots \right\} \qquad (44)$$

In the derivation of V(lnB) it is assumed that the covariances between a_1, a_2 and ln $\hat{S}_i$ are zero and that the same number of samples are tested in the inert environment as are at each stressing rate in the fatigue environment and that $K_i \simeq K_d$ where K_i is the constant K appropriate for the inert environment. The covariance between N and ln B is calculated as:

$$\mathrm{Cov}(N, \ln B) = \frac{\partial \ln B}{\partial N} V(N) = \frac{\partial \ln B}{\partial N}\frac{\partial N}{\partial N} + \frac{\partial \ln B}{\partial a_2}\frac{\partial a_2}{\partial N} + \frac{\partial \ln B}{\partial \ln \hat{S}_i}\frac{\partial \ln \hat{S}_i}{\partial N} \ldots$$

Evaluating the partial derivatives and noting that

$$\frac{\partial \ln \hat{S}_i}{\partial N} = 0 \quad \text{and}$$

$$\frac{\partial \ln \hat{S}_i}{\partial N} = \frac{\partial a_2}{\partial N} \ldots = \mathrm{Cov}(a_2, N) = \ldots$$

one gets after substituting in Eq. (A40) and Eq. (A41):

$$\mathcal{L}\,\mathcal{V}(N, \ln B) = \frac{K_0 (N+2)^2 (N-1)}{J_0 m^2 R \ln \hat{t}_0} \left[\ln B (N-1) - B \ln \hat{t}_0 - 1 - \overline{\ln \hat{t}_0} \right] \qquad (A43)$$

A.4. Static Fatigue

Similar to dynamic fatigue, the variances and covariance of N and ln B can be derived from an analysis of the static fatigue data. Assume that the static fatigue data involves J_2 different applied stresses and J_1 number of samples tested at each applied stress. By representing the failure time data by a Weibull distribution, the median failure time and its variance for the k - th applied stress level are obtained from Eqs. (A10) and (A29) as:

$$\ln \hat{t}_2 = \frac{1}{m_{0k}} \ln \ln 2 - \ln t_{0k} \qquad (A44)$$

$$\mathcal{V}(\ln \hat{t}_2) = \frac{K_{0k}}{J_1 m_{0k}^2} \qquad (A45)$$

Equations (A44) and (A45) can be written in terms of the inert strength Weibull slope since from Eq. (12):

$$m_{0k} = \frac{m_1}{N+2} \qquad (A46)$$

Thus,

$$\ln \hat{t}_2 = \frac{N+2}{m_1} \ln \ln 2 - \ln t_{0k} \qquad (A47)$$

$$\mathcal{V}(\ln \hat{t}_2) = \frac{K_0 (N+2)^2}{J_1 m_1^2} \qquad (A48)$$

where

$$K_s = \frac{m_1^2}{(N-2)^2} \frac{\Gamma\left[-\frac{2(N-2)}{m_1}\right] - \Gamma^2\left(1+\frac{N-2}{m_1}\right)}{\Gamma^2\left(1+\frac{N-2}{m_1}\right)}$$

$$- \left[\ln \ln 2 - \frac{m_1}{N-2} \ln \Gamma\left(1+\frac{N-2}{m_1}\right)\right]^2$$

Although for $\frac{m_1}{N-2} > 2$, $K_s \simeq 1.44$, this relationship is generally not true for ceramic materials. Since Eq. (A47) defines the sample parent variance, σ^2, for the median failure time, Eqs. (A7)-(A9) give for the regression coefficients a_3 and a_4 of Eq. (10):

$$V(a_3) = \frac{K_s (N-2)^2}{J_0 \; m_1^2 \; R(\ln \sigma_a)} \tag{A49}$$

$$V(a_4) = \frac{K_s (N-2)^2}{J_0 \; m_1^2} \left[1 + \frac{(\overline{\ln \sigma_a})^2}{R(\ln \sigma_a)}\right] \tag{A50}$$

$$\mathrm{Cov}(a_3, a_4) = - \frac{K_s (N-2)^2 \overline{\ln \sigma_a}}{J_0 \; m_1^2 \; R(\ln \sigma_a)} \tag{A51}$$

where $J_0 = J_1$, J_2 = total number of samples used in the static fatigue experiment

and $R(\ln \sigma_a) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \sigma_{a_j} - \overline{\ln \sigma_a})^2$.

The variances and covariance of N and ln B are calculated by applying the chain rule to Eqs. (11a) and 11b) and using Eqs. (A49 - A51) .

$$V(\ln B) = \frac{K_s (N-2)^2}{J_0 \; m_1^2 \; R(\ln \sigma_a)} \tag{A52}$$

$$V(\ln B) = \frac{K_s (N-2)^2}{J_0 m_i^2 R(\ln \sigma_a)} \left[(\ln \hat{S}_i - \overline{\ln \sigma_a})^2 + R(\ln \sigma_a) \right] + \frac{(N-2)^2 K_i}{J_1 m_i^2} \tag{A53}$$

$$Cov(N, \ln B) = \frac{K_s (N-2)^2}{J_0 m_i^2 R(\ln \sigma_a)} \left(\overline{\ln \sigma_a} - \ln \hat{S}_i \right) \tag{A54}$$

In Eqs. (A52)-(A54) it is assumed that the covariances between a_3, a_4 and $\ln \hat{S}$ are zero and that the number of inert strength samples is the same as the number of samples tested in fatigue at each applied stress.